2023-24 MATH2048: Honours Linear Algebra II

Homework 7

Due: 2023-11-06 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. **Definitions.** Two linear operators T and U on a finite-dimensional vector space V are called **simultaneously diagonalizable** if there exists an ordered basis β for V such that both $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Similarly, $A, B \in M_{n\times n}(F)$ are called simultaneously diagonalizable if there exists an invertible matrix $Q \in M_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.

   (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[T]_\beta$ and $[U]_\beta$ are simultaneously diagonalizable for any ordered basis β.

   (b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

2. (a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $TU = UT$).

   (b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

3. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \ldots, \lambda_k$. Prove that

$$\text{span}(\{x \in V : x \text{ is an eigenvector of } T\}) = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots E_{\lambda_k}.$$

4. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v.

(a) For any $w \in V$, prove that $w \in W$ if and only if there exists a polynomial $g(t)$ such that $w = g(T)(v)$.

(b) Prove that the polynomial $g(t)$ in (a) can always be chosen so that its degree is less than or equal to $\dim(W)$.

5. Let A be an $n \times n$ matrix. Prove that $\dim(\text{span}(\{I_n, A, A^2, \dots\})) \leq n$.

The following are extra recommended exercises not included in homework.

1. Let T be a diagonalizable linear operator on a finite-dimensional vector space V over F, and let $f, g \in P(F)$. Prove that $f(T)$ and $g(T)$ are simultaneously diagonalizable.

2. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(T)$-invariant for any polynomial $g(t)$.

3. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

4. Let T be a linear operator on a finite-dimensional vector space V.

   (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.

   (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

5. Use the Cayley–Hamilton theorem to prove its corollary for matrices.

6. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some polynomial $g(t)$.